On the equilibrium probabilities in deterministic flow lines with random arrivals

Woo-sung Kim and James R. Morrison
Industrial and Systems Engineering

IEEE CASE 2013 – August 20, 2013 – Madison Wisconsin, USA
Presentation Overview

- Introduction & literature review
- Preliminaries
  - Deterministic flow line
  - Channel concept
  - Delay structure in a channel
- Markov property in a channel
- Deriving equilibrium probabilities
- Special case: Single channel case with geometric distribution
- Application to semiconductor manufacturing
- Concluding remarks
Introduction

• Flow line model can be used to model data communications and production lines.

There are various studies on flow line models.

## Literature review

- Flow lines, also called tandem queue, were introduced in 1960s.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Process Time</th>
<th># of servers</th>
<th>Class of customer</th>
<th>Arrival process</th>
<th>Performance measure</th>
<th>Etc</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bound/Approximate Method</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deterministic</td>
<td>G</td>
<td>Single</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>✓</td>
<td>Upper bound, K. Park et. al. (2010)</td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>Multi class</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>✓</td>
<td>Upper bound, K. Park et. al. (2012)</td>
</tr>
<tr>
<td>Exponential</td>
<td>G</td>
<td>Single</td>
<td>M</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>- Jackson network</td>
</tr>
<tr>
<td>Random (Non-exponential)</td>
<td>2</td>
<td>Single</td>
<td>MAP</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>PH-type service time, A. Gomez-corral (2002)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Single</td>
<td>JIT</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>General service time, Muth (1973)</td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>-</td>
<td>-</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>No results</td>
</tr>
<tr>
<td><strong>Exact Method</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Deterministic</td>
<td>G</td>
<td>Single</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>✓</td>
<td>Infinite buffer before 1st process, B. Avi-Itzhak (1965)</td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>Proportional multi class</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>✓</td>
<td>Finite buffer before 1st process, Altiok and Kao (1989)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Exact decomposition, J. Morrison (2010)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Thr : Throughput
- E.P. : Equilibrium probabilities
- E.T. : Exit time

- M : Markovian
- MAP : Markovian arrival process
- JIT : Just in time
- G : General
**Literature review**

- Flow lines, also called tandem queue, were introduced in 1960s.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Process Time</th>
<th># of servers</th>
<th>Class of customer</th>
<th>Arrival process</th>
<th>Performance measure</th>
<th>Etc</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exponential</td>
<td>G</td>
<td>Single</td>
<td>M</td>
<td>√</td>
<td>√</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>G</td>
<td>Single</td>
<td>MAP</td>
<td>√</td>
<td>√</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>Random (Non-exponential)</td>
<td>G</td>
<td>Single</td>
<td>JIT</td>
<td>√</td>
<td>√</td>
<td>X</td>
</tr>
<tr>
<td></td>
<td>Deterministic</td>
<td>G</td>
<td>Single</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>Infinite buffer before 1st process</td>
</tr>
<tr>
<td></td>
<td>Deterministic</td>
<td>G</td>
<td>Proportional multi class</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>Exact decomposition</td>
</tr>
<tr>
<td></td>
<td>Deterministic</td>
<td>G</td>
<td>Proportional multi class</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>-</td>
</tr>
</tbody>
</table>

From the classic text by Altiok:
“[T]here are no known techniques to obtain measures specific to particular buffers, such as the probability distribution of the buffer contents.”
**Literature review**

- Flow lines, also called tandem queue, were introduced in 1960s.

<table>
<thead>
<tr>
<th>Approach</th>
<th>Process Time</th>
<th># of servers</th>
<th>Class of customer</th>
<th>Arrival process</th>
<th>Performance measure</th>
<th>Etc</th>
<th>Work</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>G</td>
<td>Multi class</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>Upper bound</td>
</tr>
<tr>
<td></td>
<td>Random</td>
<td>2</td>
<td>Single</td>
<td>MAP</td>
<td>√</td>
<td>X</td>
<td>General service time</td>
</tr>
<tr>
<td></td>
<td>(Non-</td>
<td>3</td>
<td>Single</td>
<td>JIT</td>
<td>√</td>
<td>X</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>exponential</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>G</td>
<td>Single</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>Infinite buffer before 1st process</td>
</tr>
<tr>
<td></td>
<td>Method</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>finite buffer before 1st process</td>
</tr>
<tr>
<td></td>
<td></td>
<td>G</td>
<td>Proportional multi class</td>
<td>G</td>
<td>X</td>
<td>X</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>G</td>
<td>Single</td>
<td>G</td>
<td>√</td>
<td>√</td>
<td>-</td>
</tr>
</tbody>
</table>
Introduction

• In here, we study deterministic flow lines consisting of arbitrary number of server and an arbitrary renewal arrival process with exact method.

![Diagram of flow line with infinite queue and servers]

• Our main results are below:
  • Identify a **Markovian property** for customer delays at each servers.
  • For **almost any discrete inter-arrival time distribution**, we obtain **Markov chain model**.
  • By solving **finite number of balance equations**, we can obtain equilibrium probabilities.
Preliminaries: Deterministic flow line

- Renewal arrival process (**discrete inter-arrival time distribution**)  
- M servers \((m_1, m_2, ..., m_M)\), deterministic service time \((\tau_1, \tau_2, ..., \tau_M)\)  
- Reliable servers, Non-idling  
- Prior to \(m_1\), there is infinite capacity of buffer.  
- Intermediate buffers can be modeled as a module with zero service time  
- Let \(B\) denote the index of the bottleneck server.  
- \(d_j(k)\) : delay in server \(m_j\) for customer \(k\), \(d_0(k)\) : delay in queue prior to \(m_1\).  
- \(\tau_j(k)\) : residence time in server \(m_j\) for customer \(k\) \((\tau_j(k) = \tau_j + d_j(k))\)
Preliminaries: Channel concept

• In [2], a deterministic flow line can be decomposed to similarly behave segment called channel.

**Definition 1.** A server $m_i$ in a deterministic flow line is called a dominating server if $\tau_i > \tau_j$ for all $j < i$. Server $m_1$ is the first dominating server.

\[
\begin{array}{ccccccc}
30 & 10 & 15 & 60 & 40 & 70 & 15 \\
\end{array}
\]

\[\Delta \quad \Delta \quad \Delta \]

$\beta(1) \quad \beta(2) \quad \beta(3)$

**Definition 2.** The $k^{th}$ channel in a deterministic flow line is the set of servers \( \{m_{\beta(k)}, m_{\beta(k)+1}, \ldots, m_{\beta(k+1)}\} \) for $1 \leq k \leq \sigma - 1$.

\[
\begin{array}{ccccccc}
30 & 10 & 15 & 60 & 40 & 70 & 15 \\
\end{array}
\]

Preliminaries : delay structure

- In [2], to calculate the delay for customer $k$, the delay for customer $k-1$, $k-2$, ..., $k-N+1$ are required where $N$ is number of servers in a channel.

- The information for 5 previous customers is required to calculate one recent customer.
Markov property in a deterministic flow line

- Employing previous results, we can obtain Markov property in a deterministic flow line.

- We derive Markov property in two steps.
  - Two successive customers arrive simultaneously.
  - Calculate delay change when inter-arrival time increases.
Markov property in a deterministic flow line

• The case that, two successive customer arrives simultaneously\((a_k=a_{k+1})\).
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously ($a_k = a_{k+1}$).

$t=0$

$a_k = a_{k+1} = 0$

Customer $k$ enter the $m_1$ directly.

<table>
<thead>
<tr>
<th>$d_0(k)$</th>
<th>$d_1(k)$</th>
<th>$d_2(k)$</th>
<th>$d_3(k)$</th>
<th>$d_4(k)$</th>
<th>$d_5(k)$</th>
<th>$d_6(k)$</th>
<th>$d_7(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k+1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that, two successive customer arrives simultaneously ($a_k = a_{k+1}$).

$$t = 0^+$$

$$a_k = a_{k+1} = 0$$

Customer $k+1$ should wait for 30 unit times.

<table>
<thead>
<tr>
<th></th>
<th>$d_0(k)$</th>
<th>$d_1(k)$</th>
<th>$d_2(k)$</th>
<th>$d_3(k)$</th>
<th>$d_4(k)$</th>
<th>$d_5(k)$</th>
<th>$d_6(k)$</th>
<th>$d_7(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k+1$</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously ($a_k = a_{k+1}$).

$t = 30^+$

$a_k = a_{k+1} = 0$

Two customers moves to the successive servers.

<table>
<thead>
<tr>
<th></th>
<th>$d_0(k)$</th>
<th>$d_1(k)$</th>
<th>$d_2(k)$</th>
<th>$d_3(k)$</th>
<th>$d_4(k)$</th>
<th>$d_5(k)$</th>
<th>$d_6(k)$</th>
<th>$d_7(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>k+1</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that, two successive customer arrives simultaneously \((a_k = a_{k+1})\).

\[
t=40^+
\]

\[
a_k = a_{k+1} = 0
\]

Customer \(k\) moves to the \(m_3\).

<table>
<thead>
<tr>
<th></th>
<th>(d_0(k))</th>
<th>(d_1(k))</th>
<th>(d_2(k))</th>
<th>(d_3(k))</th>
<th>(d_4(k))</th>
<th>(d_5(k))</th>
<th>(d_6(k))</th>
<th>(d_7(k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(d_3(k))</td>
<td>(d_4(k))</td>
<td>(d_5(k))</td>
<td>(d_6(k))</td>
<td>(d_7(k))</td>
</tr>
<tr>
<td>(k+1)</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously ($a_k=a_{k+1}$).

$t=55^+$

$$a_k=a_{k+1}=0$$

Customer $k$ moves to the $m_4$.

<table>
<thead>
<tr>
<th>$d_0(k)$</th>
<th>$d_1(k)$</th>
<th>$d_2(k)$</th>
<th>$d_3(k)$</th>
<th>$d_4(k)$</th>
<th>$d_5(k)$</th>
<th>$d_6(k)$</th>
<th>$d_7(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k+1$</td>
<td>30</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

©2013 – Woo-sung Kim and James Morrison – CASE 2013 - 17
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously \((a_k = a_{k+1})\).

\[
t = 60^+
\]

\[
a_k = a_{k+1} = 0
\]

Customer \(k+1\) moves to the \(m_2\).

<table>
<thead>
<tr>
<th></th>
<th>(d_0(k))</th>
<th>(d_1(k))</th>
<th>(d_2(k))</th>
<th>(d_3(k))</th>
<th>(d_4(k))</th>
<th>(d_5(k))</th>
<th>(d_6(k))</th>
<th>(d_7(k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(k+1)</td>
<td>30</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously \( a_k = a_{k+1} \).

\[
t = 70^+ \\
\begin{align*}
  a_k &= a_{k+1} = 0
\end{align*}
\]

Customer \( k+1 \) moves to the \( m_3 \).

<table>
<thead>
<tr>
<th></th>
<th>( d_0(k) )</th>
<th>( d_1(k) )</th>
<th>( d_2(k) )</th>
<th>( d_3(k) )</th>
<th>( d_4(k) )</th>
<th>( d_5(k) )</th>
<th>( d_6(k) )</th>
<th>( d_7(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k+1 )</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously ($a_k = a_{k+1}$).

$t = 85^+$

$a_k = a_{k+1} = 0$

Customer $k+1$ complete its service in $m_3$, but it cannot proceed.

<table>
<thead>
<tr>
<th></th>
<th>$d_0(k)$</th>
<th>$d_1(k)$</th>
<th>$d_2(k)$</th>
<th>$d_3(k)$</th>
<th>$d_4(k)$</th>
<th>$d_5(k)$</th>
<th>$d_6(k)$</th>
<th>$d_7(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k+1$</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- The case that two successive customer arrives simultaneously \(a_k = a_{k+1}\).

\[ t = 115^+ \]

\[ a_k = a_{k+1} = 0 \]

After customer \( k \) complete its service in \( m_4 \), customer \( k+1 \) enter the \( m_4 \).

<table>
<thead>
<tr>
<th>( d_0(k) )</th>
<th>( d_1(k) )</th>
<th>( d_2(k) )</th>
<th>( d_3(k) )</th>
<th>( d_4(k) )</th>
<th>( d_5(k) )</th>
<th>( d_6(k) )</th>
<th>( d_7(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( d_4(k) )</td>
<td>( d_5(k) )</td>
<td>( d_6(k) )</td>
</tr>
<tr>
<td>( k+1 )</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Markov property in a deterministic flow line

- Given delay of customer \( k \), we can calculate delay of customer \( k+1 \) when \( a_k = a_{k+1} \).

<table>
<thead>
<tr>
<th></th>
<th>( d_0(k) )</th>
<th>( d_1(k) )</th>
<th>( d_2(k) )</th>
<th>( d_3(k) )</th>
<th>( d_4(k) )</th>
<th>( d_5(k) )</th>
<th>( d_6(k) )</th>
<th>( d_7(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( k+1 )</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- **Lemma 3.3**: When customer \( k \) and \( k+1 \) arrives simultaneously (\( a_k = a_{k+1} \)), the delay of customer \( k+1 \) can be calculated by the delay of customer \( k \).

\[ a_{k+1} - a_k = 0 \]

\{0,0,0,0,0,0,0,0\}

Customer \( k \)

\{30,0,0,30,0,10,0,0\}

Customer \( k+1 \)
Markov property in a deterministic flow line

- When inter-arrival time increases, we can calculate delay change in a deterministic flow line.

- **Lemma 3.4:** Delay of customer $k+1$ can be calculated as the inter-arrival time increase when delay of customer $k$ is given.
Lemma 3.3: The delay in a channel for simultaneous arrivals. Consider a particular channel $\alpha$ and the case where both customers $k$ and $k + 1$ arrive to the deterministic flow line simultaneously ($a_{k+1} = a_k$). Given the server delays for the $k^{th}$ customer in the channel, the server delays for the $k + 1^{th}$ customer in the channel obey the following recursions.

If $j^* > \beta(\alpha)$,
\[
d_i(k + 1) = 0 \quad \text{for } i = \beta(\alpha), \ldots, j^* - 2.
\]

If $\tau_{\beta(\alpha)} \geq \tau_{j^*} + d_{j^*}(k)$
\[
d_{j^*-1}(k + 1) = 0
\]
\[
d_{j^*}(k + 1) = d_{j^*}(k) + d_{j^*+1}(k) + \tau_{j^*+1} - \tau_{\beta(\alpha)}
\]
\[
d_l(k + 1) = d_{l+1}(k) + \tau_{l+1} - \tau_l
\]
for $l = j^* + 1, \ldots, \beta(\alpha + 1) - 1$.

If $\tau_{\beta(\alpha)} < \tau_{j^*} + d_{j^*}(k)$
\[
d_{j^*-1}(k + 1) = \tau_{j^*} + d_{j^*}(k) - \tau_{\beta(\alpha)}
\]
\[
d_l(k + 1) = d_{l+1}(k) + \tau_{l+1} - \tau_l
\]
for $l = j^*, \ldots, \beta(\alpha + 1) - 1$.

If $j^* = \beta(\alpha)$,
\[
d_l(k + 1) = d_{l+1}(k) + \tau_{l+1} - \tau_l
\]
for $l = j^*, \ldots, \beta(\alpha + 1) - 1$.

Lemma 3.4: Delay change in a deterministic flow line as the interarrival time increases. Let the delays for the $k^{th}$ and $k + 1^{th}$ customers be given as $d(k)$ and $d(k + 1)$. Their arrival times are $a_k$ and $a_{k+1}$. For a positive real number $\delta \leq d_F(k+1)(k + 1)$, if $a_{k+1}$ is changed to $a_{k+1} + \delta$, all other delays except for $d_F(k+1)(k + 1)$ remain unchanged. $d_F(k+1)(k + 1)$ becomes $d_F(k+1)(k + 1) - \delta$. 
Deriving equilibrium probabilities

- Assume that the inter-arrival time distribution is discrete, we can obtain Markov chain.

**Theorem 3.3: Markov chain for discrete time flow lines.** Consider a discrete-time flow line with renewal arrival process. The process \( \{d(k)\}_{k=1}^{\infty} \) is a discrete-time, time-homogeneous Markov chain.

- Equilibrium probabilities can be obtained by solving the balance equations.
Deriving equilibrium probabilities

• There are an infinite number of states

These states can be obtained directly.

• **Theorem 3.5**: When total delay is sufficiently big, the equilibrium probabilities for those states can be obtained directly from the results on GI/D/1 queue.

• By the theorem, we can finite number of balance equations.
Special case: single channel

- As a special case, we consider a deterministic flow line consisting of a single channel with geometric inter-arrival time distribution \( P\{a_k-a_{k-1}=k\}=(1-p)^{k-1}p \).
- For single channel, the server delays can be obtained from \( Y(k) \) where \( Y(k)=d_1(k)+...+d_{B-1}(k) \).

\[
1 - \sum_{k=\tau_g}^{\infty} (1-p)^k p = \tau_g - \tau_1
\]

< Transition probabilities >

- Two dimensional transition diagram can be derived.
Special case: single channel

- With this structure and our method, we can derive equilibrium probabilities.

- The probabilities in each region can be obtained separately.
Special case: single channel

• With this structure and our method, we can derive equilibrium probabilities by each region.

\[
\begin{align*}
\Pi_{0,0} &= \frac{3p-1}{p-1} \\
\Pi_{0,2} &= \left(\frac{p}{(1-p)^2} - p\right)\frac{3p-1}{p-1} \\
\Pi_{1,2} &= \left(\frac{p^2}{1-p} + (1-p)p^2\right)\frac{3p-1}{p-1} \\
\Pi_{1,1} &= p\frac{3p-1}{p-1} \\
\Pi_{2,2} &= p^2\frac{3p-1}{p-1}
\end{align*}
\]
Special case: single channel

- For remaining states, we can derive probability generating function.

\[ X(z) = \left[ \frac{p^3(-5z - 3z^2 - z^3 + p^3(z^2 + z + 1) - p^2(4 + 4z + 4z^2 + z^3) + p(7 + 7z + 6z^2 + 2z^3))}{(p-1)^2(-1 + p(1 + z + z^2))} \right] \frac{3p-1}{p-1} \]
Application to semiconductor manufacturing

- In semiconductor wafer manufacturing, clustered photolithography tools are the most expensive tools.

The tool can be modeled by flow line model.
Application to semiconductor manufacturing

• Modeled by deterministic flow line model.

• Serializing for analytic tractability

• The serialized flow line model provide throughput and cycle time estimates within 0.5% and 3% of actual ([2])

Application to semiconductor manufacturing

• Considering setup, delay distribution is required to calculate production rate ([3]).

• **Proposition:** The inter-departure times satisfy the following recursion,

\[
C_M(k+1) - C_M(k) = \begin{cases} 
\tau_B \\
\tau_B + \{\tau_B - \tau_B + \sum_{i=1}^{S_{II}} \tau_i + \sum_{i=S_{II}+1}^{B-1} d_i(k)\}^+ 
\end{cases}
\]

with \( p_{II} \)

with \( 1-p_{II} \)

• In [3], the production rate is only obtained when setup server is in the last channel.

• By our results, this restriction can be relaxed.

Concluding remarks

• Concluding remark
  • We identify a Markov property for the customer delay in the servers
  • When inter-arrival time is discrete time distribution, the delays can be modeled as a multi-dimensional discrete time, time-homogeneous Markov chain.
  • Equilibrium probabilities can be numerically obtained by solving finite number of balance equations.

• Future direction
  • Find specific model which possess exact closed form expression for the equilibrium probabilities.
  • Continuous time case.