On Equilibrium Probabilities in a Class of Two Station Closed Queueing Networks

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Abstract: Due to the finite state space of closed exponential queueing networks, one can obtain the equilibrium probability distribution by directly solving the global balance equations. However, as the number of trapped customers increases, the state space grows and it becomes practically impossible to obtain a solution. Alternatives include performance bounds, approximations or simulation, see [1]-[3]. These techniques may provide efficient tools to analyze the network, but cannot provide complete steady state behavior. By focusing our attention on a class of two station closed reentrant queueing networks, we obtain closed form expressions for the equilibrium probability distribution that are computationally independent of the number of trapped customers. The computational complexity depends only on the network structure. By considering the last buffer first served (LBFS) policy, we can reduce the state space to a three dimension rectangle whose height increases with the number of trapped customers. By recognizing a sense of causality in the balance equations, we are able to employ z-transform techniques to obtain an explicit solution for the equilibrium probabilities. Several examples, including the closed Lu-Kumar network under LBFS are studied to demonstrate the approach. The networks identified represent the only class of non-product form queueing networks which, to our knowledge, possess an explicit equilibrium probability distribution.

Keywords: closed queueing networks, equilibrium probabilities, closed form solution, buffer priority policy

1. INTRODUCTION

A natural class of models for modern systems such as semiconductor wafer fabricators and service systems is multiclass queueing networks. Unfortunately, excepting the well known product form networks, there are few multiclass queueing networks that admit explicit solutions for their equilibrium probabilities. Due to this intractability, many network analyses have focused on performance bounds, approximations or simulation, see [1]-[3]. These techniques may provide efficient tools to analyze the network, but cannot provide complete steady state behavior. Closed multiclass queueing networks, in which a fixed number of N customers circulate endlessly, have been used to model automated material handling systems (AMHS) [4], pull production systems [5] and internet communication protocols [6].

Closed exponential queueing networks are more tractable than their open counterparts in the sense that they have a finite state space. Thus, the global balance equations [7] can be solved to obtain the equilibrium probability distribution (EPD). However, the state space dramatically increases as the number of trapped customers N increases, so that it is no longer computationally possible to obtain the EPD.

In this paper, we study a class of two station closed queueing networks under the last buffer first served priority policy. These networks are generalized version of the Lu-Kumar network in [8]. We introduce a technique to obtain the equilibrium probabilities. Independent of the number of buffers, we can reduce the state space to four dimensions under LBFS. First, we obtain a matrix form for the global balance equations. Using linear algebra techniques, we next obtain the z-transform of the EPD. By inverting the z-transform, we can characterize the equilibrium probabilities of the system. To verify our technique, we obtain equilibrium probabilities for the Lu-Kumar network under LBFS.

The remainder of this paper is organized as follow. In section 2, we describe the system and define variables. In section 3, we introduce the technique to obtain the EPD of the system. We first characterize the global balance equations and derive the z-transform. Based on the z-transform technique, we can obtain the equilibrium probability distribution by inverting the z-transform. In section 4, we provide examples of our technique. Concluding remarks are presented in section 5.

2. SYSTEM DESCRIPTION

We analyze a class of two station closed reentrant queueing networks. These networks consist of two stations, labeled $\sigma_1$ and $\sigma_2$, and $m+n$ buffers, labeled $b_1,b_2,\ldots,b_{m+n}$. Buffers $b_j$ to $b_m$ are served at station $\sigma_1$ and buffers $b_{m+1}$ to $b_{m+n}$ are served at station $\sigma_2$. Figure 1 depicts such a system. The service time for a customer in buffer $b_j$ is exponentially distributed with rate $\mu_j$, and all service times are independent. Because the system is
a closed queueing network with N trapped customers, there are no external arrivals or departures. When a customer in buffer $b_i$ finishes its service, the customer next moves to buffer $b_{i+1}$, unless $i=n+m$, in which case the customer queues in buffer $b_1$. A station can serve at most one customer at a time, so a scheduling policy is required. A scheduling policy dictates which non-empty buffer a station should serve. In this paper, we use the preempt-resume last buffer first served (LBFS) policy. Under LBFS, buffer $b_i$ has priority over buffer $b_j$ when $j<i$. Therefore, in steady state, there can be at most one customer in $b_2$ to $b_m$ and at most one in $b_{m+2}$ to $b_{m+n}$.

### 3. Z-TRANSFORM AND EQUILIBRIUM PROBABILITIES

In this section, we introduce the procedure to obtain a general explicit closed form solution for the equilibrium probabilities in our network. First we define the state space. Under the LBFS policy, at most one customer can exist in $b_2$ to $b_m$ and at most one in $b_{m+2}$ to $b_{m+n}$ in steady state. Using this fact, we can reduce the state space to four dimensions. Based on the reduced state space, we derive the matrix form of the global balance equations (GBEs). The GBEs of the system have a recursive structure. Therefore, if we know the initial conditions, we can express all probabilities as initial conditions. After that we can obtain the $z$-transform of the system. Inverting it and using the remaining equations, we can obtain the equilibrium probability distribution.

#### 3.1 State space and matrix form of GBEs

Let $s(t) = (w(t), x(t), y(t), z(t))$ denote the state of the system at time $t$. Let $w(t)$ and $y(t)$ denote the number of customers in buffers $b_2$ and $b_{m+1}$ at time $t$, respectively. Let $x(t)$ and $z(t)$ denote the buffer index if there is a customer in $b_2$ to $b_m$ and $b_{m+2}$ to $b_{m+n}$ at time $t$, respectively. If there is no customer in $b_2$ to $b_m$ at time $t$, then $x(t)$ equals 0. Similarly for $z(t)$. For example, consider the state $\{4,3,2,2\}$. This state means that there are four customers in $b_1$. The second value, three, means that one customer is in $b_3$. Because we employ the LBFS policy, the customer in $b_3$ is being served by station 1. Similarly, the third number, two, means that there are two customers in $b_2$. The last number means that there is one customer in $b_{m+1}$. Because the policy is stationary, sampling at times when real or virtual services occur gives a discrete-time, finite state, time-homogeneous Markov chain [8]. We define the steady state probability as

$$X_{i\gamma}(k) = \lim_{t \to \infty} \text{Prob}(w(t) = N \cdot k + \delta, x(t) = i, y(t) = k - \gamma, z(t) = j),$$

where $\delta = \gamma = 1$ if $i = j = 0$, $\delta = 2, \gamma = 1$ if $i = 0, j = 0$, $\delta = 1, \gamma = 0$ if $i = 0, j \neq 0$, otherwise $\delta = 2$ and $\gamma = 0$. When $k$ equals -1, there are $m$ probabilities, $X_{i00}(1)$, $X_{i20}(1)$, $X_{i30}(1)$ ... $X_{in0}(1)$. We call these the initial condition probabilities. And we also define $X[-1]$ as the initial condition vector.

All $R_i$’s are $m$-$m$ matrices. Above, the 0 entry indicates an m-$m$ matrix in which all entries are 0. Note that the matrix equation (3) has a recursive structure. That is, if we know $X[-1]$, then we can obtain $X[0]$. For simplicity, we define another matrix $R$ to represent this relationship. We can express $X[0]$ in terms of $X[1]$ as
Similarly, we can obtain the probabilities for the $k=1$ case assuming that we know the initial conditions. The equations are given below:

\[
X[0] = RX[-1]. \tag{4}
\]

Using (4) and (5). In a similar way, we define the matrices. Note that assuming we know the initial conditions, we can obtain $X[1]$ using (4) and (5). In a similar way, we define the matrix $S$ as

\[
X[1] = SX[-1]. \tag{6}
\]

Finally, the equation for general $K$ are

\[
X[k] = \begin{bmatrix} X_0[k] & 0 & A_0 & 0 & \ldots & 0 & X_n[k] \\ X_1[k] & 0 & A_1 & 0 & \ldots & 0 & X_n[k] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{n-1}[k] & 0 & 0 & \ldots & 0 & A_{n-1} & X_n[k] \\ X_n[k] & 0 & 0 & \ldots & 0 & 0 & X_n[k] \end{bmatrix} + \begin{bmatrix} 0 & B_1 & 0 & \ldots & 0 & X_0[k-1] \\ 0 & 0 & B_1 & 0 & \ldots & 0 & X_0[k-1] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & B_{n-1} & X_0[k-1] \\ B_n & 0 & 0 & \ldots & 0 & 0 & X_0[k-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & X_0[k-2] \\ 0 & 0 & 0 & \ldots & 0 & X_0[k-2] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & X_0[k-2] \\ C & 0 & 0 & \ldots & 0 & X_0[k-2] \end{bmatrix} \tag{7}
\]

where

\[
A_i = \begin{bmatrix} \mu_1 + \mu_{\text{init}} & 0 & \ldots & 0 \\ \mu_{\text{mini}} & \mu_2 + \mu_{\text{init}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{\text{mini}} & \mu_{\text{mini}} & \ldots & \mu_{\text{ini}} + \mu_{\text{init}} \end{bmatrix}, \quad \text{for } i = 1, 2, \ldots, n-1
\]

\[
B_i = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad \text{for } i = 1, 2, \ldots, n-1
\]
\[(I-A)z^2-Bz-C)X(z)=[(I-A)z^2RX[1]-1]+z(I-A)SX[-1]-zBRX[-1].\]  

Multiplying by the inverse of \([(I-A)z^2-Bz-C]\) on both sides of equation (11), the z-transform of the system can be obtained. Defining the matrix D as \([(I-A)z^2-Bz-C]\), we can express \(X(z)\):

\[X(z)=\frac{\text{adj } D}{\det D}[(I-A)z^2R+z(I-A)S-zBR]X[-1].\]  

The notation \(\det D\) means the determinant of the matrix D, and \(\text{adj } D\) is the mn × mn matrix whose \((i,j)\)-entry is \((-1)^{i+j}\) det \(D_{ij}\). The matrix \(D_{ij}\) denotes the sub-matrix of D formed by deleting row i and column j. To characterize the z-transform, we investigate det D.

In \([(I-A)z^2-Bz-C], z^2s are multiplied in each row. Therefore, det D is a polynomial in which the degree of z is 2mn. Thus, det D can be written as:

\[
\det D = \sum_{i=0}^{2mn} a_i z^i. 
\]  

Next, we investigate the numerator of (12). The matrix which is obtained by \(\text{adj } D\) multiplied by \([(I-A)z^2R+z(I-A)S-zBR]\) is an mn × mn matrix. Each element in \(\text{adj } D\) contains a polynomial of degree 2mn−2 or less because each is obtained by deleting one row and column of the matrix D. Because the maximum degree of z of each element of \([(I-A)z^2R+z(I-A)S-zBR]\) is 2, each element of the numerator matrix of (12) also contains a polynomial with degree 2mn or less. So we can represent \(X(z)\) as below:

\[X(z)=C_z X[-1]\]  

We rewrite the matrix equation for \(k \geq 2\) as

\[(I-A)X[k]=BX[k-1]+C X[k-2], \text{ for } k=2, \ldots, N-1.\]  

We can obtain \(X[k]\) assuming that we know \(X[k-1]\) and \(X[k-2]\) by multiplying the inverse of \((I-A)\) to both side. The equation for \(X[k]\) is given below:

\[X[k]=(I-A)^kBX[k-1]+(I-A)^kC X[k-2], \text{ for } k=2, \ldots, N-1\]  

We again see that the matrix equations have a recursive forms. Therefore, if we know the initial conditions, we can express all equilibrium probabilities in terms of the initial conditions.

### 3.2 Z-transform and equilibrium probabilities.

In this section, we obtain the z-transform of the equilibrium probabilities. After that, we obtain the probabilities by inverting the z-transform. First, taking the z-transform of equation (9), we obtain

\[\{(I-A)z^2-Bz-C\}X(z)=[(I-A)z^2X[0]-1]+z(I-A)X[1]-zBRX[0].\]  

In (10), we can express \(X[0]\) and \(X[1]\) in terms of \(X[-1]\) using (4) and (6):

\[\{(I-A)z^2-Bz-C\}X(z)=\{(I-A)z^2RX[1]-1\}+z(I-A)SX[-1]-zBRX[-1].\]  

Therefore, if we know each \(C_{ij,k}\), we can express the z-transforms in terms of only the initial conditions. By
inverting the above z-transform, we can represent each probability in terms of m initial condition probabilities.

Using partial fraction expansion, we can rewrite $C_{ij,k}(z)$ below as

$$
C_{ij,k}(z) = \sum_{b_{ijkl}} \frac{\sum_{k=0}^{\infty} b_{ijkl} z^k}{(z - p_{ijkl})^n}
$$

where $\sum b_{ijkl} = 2m$.

The poles of $C_{ij,k}(z)$ are the roots of determinant $D$. Therefore, if we know $\beta_{ij,k,l,m}$, we have obtained the z-transform in a form which can easily be inverted.

There is a standard procedure to obtain $\beta_{ij,k,l,m}$ recursively. We introduce how to obtain $\beta_{ij,k,l,m}$.

First, by plugging $z=0$ into (16) we can obtain $\beta_{i,j,k,l,m}$. Next, multiplying (16) by $(z-p_{ijkl})a_1$ and evaluating at $z=p_{ijkl}$, then we can obtain the $\beta_{ijkl,1,2}$ coefficient of $(z-p_{ijkl})a_3$. Then, we can obtain the coefficient of $z^{n-1}$ using the backward recursion.

$$
\beta_{ijkl,n-1} = \frac{d^n}{dz^n} \left[ \frac{C_{ijkl}(z)}{z^{n-1}} \right]_{z=p_{ijkl}} - \frac{1}{n!} \sum_{a=n-1}^{n} \beta_{ijkl,a}.
$$

First put $r=1$ into the equation to obtain $\beta_{ijkl,1,1}$. Then we can obtain the other coefficients recursively. Next, we invert equation (15), so that we can represent $X_{ij}(k)$ as a linear combination of the inverse of the $C_{ijk}(z)$'s and the constants of $X[-1]$. We thus define $C_{ij,k}(n)$ as the inverse of $C_{ij,k}(z)$.

$$
C_{ij,k}(n) = \frac{\sum_{b_{ijkl}} b_{ijkl} z^k}{(z - p_{ijkl})^n}
$$

where $\sum b_{ijkl} = 2m$.

Using these equations, we obtain the elements of $X[-1]$. With this approach, we can obtain the equilibrium probabilities of the system.

4. SPECIAL CASE: $m=2$, $n=2$ AND SAME SERVICE RATE
In this section, we provide an example of our technique. In the example, we assume each station has two buffers and customer in each buffer is served with same rate \( \mu \). The results are already in \([10]\). First, we can obtain matrices below:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  \( (23) \)

Next, to obtain the z-transform of the system, we can obtain \([-z(A)z^2-Bz-C]\) below:

\[
[-z(A)z^2-Bz-C] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-2z & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  \( (24) \)

Plugging above matrices into the equation (11), we can obtain z-transform of the system. The z-transform is below:

\[
X_{00}(z) = \frac{x_{20}(1)[z^2-z^2]}{z^3-7z^2+7z-1} + \frac{zX_{00}(-1)[z^2-z^2+2z^2]}{z^3-7z^2+7z-1} + \frac{x_{02}(1)[z^2-z^2]}{z^3-7z^2+7z-1} + \frac{zX_{02}(-1)[z^2-z^2]}{z^3-7z^2+7z-1} + \frac{x_{22}(1)[z^2-z^2]}{z^3-7z^2+7z-1} + \frac{zX_{22}(-1)[z^2-z^2]}{z^3-7z^2+7z-1}
\]  \( (25) \)

We can see that we can express z-transform of the system in terms of initial probabilities. Using the remaining equations, we can obtain the z-transform and equilibrium probabilities of the system.

4. CONCLUSION

We considered a class of closed exponential reentrant queueing networks. Under the LBFS, the state space can be reduced to four dimensions. First we investigate the global balance equations and express them in a matrix form. Then using linear algebra and z-transform techniques, we obtain the equilibrium probabilities for this class of networks. There are few networks with explicit expressions for their equilibrium probabilities. This class of networks is a new class of closed network that can be characterized completely. To demonstrate the technique, we applied our approach to the closed Lu-Kumar network under LBFS.

REFERENCES


